Inequalities - Maximum and Minimum

1. From A.M. ≥ G.M. , $\sqrt[n]{x_1x_2...x_n} \le \dfrac{x_1 + x_2 + ... + x_n}{n} = \dfrac{C}{n}$ $\Rightarrow x_1x_2...x_n \le \left(\dfrac{C}{n}\right)^n$

Hence, the product $x_1x_2\ldots x_n$ does not exceed $\left(\dfrac{C}{n}\right)^n$ and reaches it iff $x_1 = x_2 = ... = x_n = \dfrac{C}{n}$.

2. From A.M. ≥ G.M. , $\dfrac{x_1 + x_2 + ... + x_n}{n} \ge \sqrt[n]{x_1x_2...x_n}$ $\Rightarrow x_1 + x_2 + ... + x_n \ge n\sqrt[n]{C}$

Equalities holds $\Leftrightarrow$ $x_1 = x_2 = ... = x_n$

$\therefore$ The sum $x_1 + x_2 + \ldots + x_n$ attains the least value $n\sqrt[n]{C}$ when: $x_1 = x_2 = ... = x_n = \sqrt[n]{C}$.

3. First let us assume that $\mu_i \in \mathbf{N}$. We have by A.M. ≥ G.M. ,

$$\sqrt[\mu_1+\mu_2+....+\mu_n]{\left(\frac{x_1}{\mu_1}\right)^{\mu_1}\left(\frac{x_2}{\mu_2}\right)^{\mu_2}...\left(\frac{x_n}{\mu_n}\right)^{\mu_n}} \le \frac{\mu_1\left(\frac{x_1}{\mu_1}\right) + \mu_2\left(\frac{x_2}{\mu_2}\right) + ... + \mu_n\left(\frac{x_n}{\mu_n}\right)}{\mu_1 + \mu_2 + ... + \mu_n} = \frac{C}{\mu_1 + \mu_2 + ... + \mu_n}$$

$$\therefore \quad x_1^{\mu_1}x_2^{\mu_2}...x_n^{\mu_n} \le \left(\frac{C}{\mu_1 + \mu_2 + ... + \mu_n}\right)^{\mu_1+\mu_2+....+\mu_n} \times \left(\mu_1^{\mu_1}\mu_2^{\mu_2}....\mu_n^{\mu}\right)$$

Equalities holds $\Leftrightarrow$ $\dfrac{x_1}{\mu_1} = \dfrac{x_2}{\mu_2} = ... = \dfrac{x_n}{\mu_n} = \dfrac{x_1 + x_2 + ... + x_n}{\mu_1 + \mu_2 + ... + \mu_n} = \dfrac{C}{\mu_1 + \mu_2 + ... + \mu_n}$.

Now, let us assume that $\mu_i \in \mathbf{Q}$.

Reducing them to a common denominator, we put $\mu_i = \dfrac{\lambda_i}{\mu}$, where $\lambda_i , \mu \in \mathbf{N}$.

Since $x_1^{\mu_1}x_2^{\mu_2}...x_n^{\mu_n} \le \sqrt[\mu]{x_1^{\lambda_1}x_2^{\lambda_2}...\,x_n^{\lambda_n}}$, the greatest value is reached by the product $x_1^{\mu_1}x_2^{\mu_2}...\,x_n^{\mu_n}$

simultaneously with the product $x_1^{\lambda_1}x_2^{\lambda_2}...\,x_n^{\lambda_n}$, where $\lambda_i \in \mathbf{N}$.

As followed from the above proof, it happens iff $\dfrac{x_1}{\mu_1} = \dfrac{x_2}{\mu_2} = ... = \dfrac{x_n}{\mu_n}$.

$\therefore$ If $x_i > 0$ and $x_1 + x_2 + \ldots + x_n = C$, then the product $x_1^{\mu_1}x_2^{\mu_2}...\,x_n^{\mu_n}$ $(\mu_i \in \mathbf{Q} , \mu_i > 0)$

attains the greatest value iff $\dfrac{x_1}{\mu_1} = \dfrac{x_2}{\mu_2} = ... = \dfrac{x_n}{\mu_n}$.

4. From A.M. ≥ G.M. , $\sqrt[n]{(a_1x_1)(a_2x_2)...(a_nx_n)} \le \dfrac{a_1x_1 + a_2x_2 + ... + a_nx_n}{n} = \dfrac{C}{n}$.

It follows that the product $(a_1x_1)(a_2x_2)...(a_nx_n)$ reaches the maximum $\Leftrightarrow$ $a_1x_1 = a_2x_2 = ... = a_nx_n$

But since $(a_1x_1)(a_2x_2)...(a_nx_n) = (a_1a_2...a_n)(x_1x_2...x_n)$, therefore the product $x_1x_2\ldots x_n$ reaches the greatest value when : $a_1x_1 + a_2x_2 + ... + a_nx_n = \frac{C}{n}$.

5. Put $a_ix_i^{\lambda_i} = y_i$, $x_i = \left(\frac{y_i}{a_i}\right)^{1/\lambda_i}$ where $i = 1,2, \ldots, n$.

and $y_1 + y_2 + \ldots + y_n = C$.

Furthermore, $x_1^{\mu_1}x_2^{\mu_2}\ldots x_n^{\mu_n} = \left(\frac{y_1}{a_1}\right)^{\mu_1/\lambda_1}\left(\frac{y_2}{a_2}\right)^{\mu_2/\lambda_2}\ldots\left(\frac{y_n}{a_n}\right)^{\mu_n/\lambda_n}$. The problem is reduced to finding out

when the product $y_1^{\mu_1/\lambda_1}y_2^{\mu_2/\lambda_2}\ldots y_n^{\mu_n/\lambda_n}$ takes on the greatest value if $y_1 + y_2 + \ldots + y_n = C$.

By No. 3, we see that it will take place $\Leftrightarrow$ $\frac{y_1}{\mu_1/\lambda_1} = \frac{y_2}{\mu_2/\lambda_2} = \ldots = \frac{y_n}{\mu_n/\lambda_n}$.

Thus if $a_1x_1^{\lambda_1} + a_2x_2^{\lambda_2} + ... + a_nx_n^{\lambda_n} = C$,

then the product $x_1^{\mu_1}x_2^{\mu_2}\ldots x_n^{\mu_n}$ is maximum $\Leftrightarrow$ $\frac{\lambda_1a_1x_1^{\lambda_1}}{\mu_1} = \frac{\lambda_2a_2x_2^{\lambda_2}}{\mu_2} = \ldots = \frac{\lambda_na_nx_n^{\lambda_n}}{\mu_n}$.

6. Put $a_ix_i^{\mu_i} = y_i$, $x_i = \left(\frac{y_i}{a_i}\right)^{1/\mu_i}$ where $i = 1,2, \ldots, n$. The problem is reduced to finding the least value of $y_1 + y_2 + \ldots + y_n$ if $y_1^{\lambda_1/\mu_1}y_2^{\lambda_2/\mu_2}\ldots y_n^{\lambda_n/\mu_n} = C_1$, where C_1 is a new constant.

Since λ_i/μ_i $(i = 1, 2, \ldots, n)$ are rational, we put $\frac{\lambda_i}{\mu_i} = \frac{\alpha_i}{N}$, where N is the common denominator and α_i, N are positive integers.

The problems becomes to finding out when $y_1 + y_2 + \ldots + y_n$ attains the least value if $y_1^{\alpha_1}y_2^{\alpha_2}\ldots y_n^{\alpha_n} = C_2$.

Finally, we put $y_i = \alpha_i u_i$ where $i = 1,2, \ldots, n$ and obtain the following problem :
under what condition does $\alpha_1u_1 + \alpha_2u_2 + \ldots + \alpha_nu_n$ attain the least value if $u_1^{\alpha_1}u_2^{\alpha_2}\ldots u_n^{\alpha_n} = C_3$.

But $\frac{\alpha_1u_1 + \alpha_2u_2 + \ldots + \alpha_nu_n}{\alpha_1 + \alpha_2 + \ldots + \alpha_n} \geq \sqrt[\alpha_1+\alpha_2+\ldots+\alpha_n]{u_1^{\alpha_1} + u_2^{\alpha_2} + \ldots + u_n^{\alpha_n}} = \sqrt[\alpha_1+\alpha_2+\ldots+\alpha_n]{C_3}$.

Hence $\alpha_1u_1 + \alpha_2u_2 + \ldots + \alpha_nu_n$ attains the least value $\Leftrightarrow$ $u_1 = u_2 = \ldots = u_n$.

$a_1x_1^{\mu_1} + a_2x_2^{\mu_2} + ... + a_nx_n^{\mu_n}$ attains the least value when: $\frac{x_1^{\mu_1}}{\frac{\lambda_1}{a_1\mu_1}} = \frac{x_2^{\mu_2}}{\frac{\lambda_2}{a_2\mu_2}} = ... = \frac{x_n^{\mu_n}}{\frac{\lambda_n}{a_n\mu_n}}$.

7. Since $x + y + z = \frac{\pi}{2}$, $0 \leq x, y, z \leq \frac{\pi}{2}$,

tan x tan y + tan x tan z + tan y tan z = tan x tan y + (tan x + tan y) tan z

= tan x tan y + (tan x + tan y) tan [π/2 – (x + y)] = tan x tan y + (tan x + tan y) cot (x + y)

= tan x tan y + (tan x + tan y) [1/ tan(x + y)] = tan x tan y + 1 – tan x tan y = 1

Thus the sum of the three quantities tan x tan y , tan x tan z , tan y tan z is constant.

Therefore the product of these quantities, that is, $\tan^2 x \tan^2 y \tan^2 z$ reaches the greatest value if

tan x tan y = tan x tan z = tan y tan z .

We have tan x tan y tan z reaches the greatest value if tan x = tan y = tan z , or x = y = z = π/6 .

8. **(Method 1)**

$$\frac{1}{n+1}+\frac{1}{n+2}+....+\frac{1}{3n+1}=\left(\frac{1}{n+1}+\frac{1}{3n+1}\right)+\left(\frac{1}{n+2}+\frac{1}{3n}\right)+..+\frac{1}{2n+1}$$

$$=\frac{4n+2}{(n+1)(3n+1)}+\frac{4n+2}{(n+2)(3n)}+...+\frac{4n+2}{2(2n+1)^2}=(4n+2)\left[\frac{1}{(n+1)(3n+1)}+\frac{1}{(n+2)(3n)}+...+\frac{1}{2(2n+1)^2}\right]$$

$$>(4n+2)\left[\frac{1}{(2n+1)^2}+...+\frac{1}{(2n+1)^2}+\frac{1}{2(2n+1)^2}\right]=(4n+2)\left[\frac{n}{(2n+1)^2}+\frac{1}{2(2n+1)^2}\right]=(4n+2)\left[\frac{2n+1}{2(2n+1)^2}\right]$$

= 1

Note : For the inequality sign in the above proof, observe that :

$(n + 1 – i) (3n + 1 – i) = [(2n + 1) – (n – i)] [(2n + 1) + (n – i)] = (2n + 1)^2 – (n – i)^2 < (2n + 1)^2$, i = 1 to n .

(Method 2)

Use Arithmetic mean > Harmonic mean.

$$\frac{\frac{1}{n+1}+\frac{1}{n+2}+....+\frac{1}{3n+1}}{2n+1}>\frac{2n+1}{(n+1)+(n+2)+...+(3n+1)}=\frac{1}{2n+1} \quad .$$

9. Let $a_1 + a_2 + … + a_p = n = pq + r$, we like to prove that the min. of $a_1! \, a_2 ! … a_p !$ is $(q!)^{p-r} [(q + 1)!]^r$.

First we prove that : If a – 1 > b then a! b! > (a – 1)! (b + 1)! …. (1)

Proof : a! b! – (a – 1)! (b + 1)! = (a – 1)! b! [a – (b + 1)] = (a – 1)! b! [(a – 1) – b] > 0 . Result follows.

Without lost of generality, we assume : $a_1 \le a_2 \le … \le a_p$.

Case 1 If $a_p – 1 = a_1$, then $a_1 = a_2 = … = a_m = a_{m+1} – 1 = a_{m+2} – 1 = … = a_p – 1$.

But $a_1 + a_2 + … + a_p = n = pq + r = (p – r)q + r(q + 1)$

∴ m = p – r and $a_1 = a_2 = … = a_m = q$, $a_{m+1} = a_{m+2} = … = a_p = q + 1$

∴ $a_1! \, a_2 ! …a_p ! = (q! \, q! …q!) (q + 1)! (q + 1)! … (q + 1)! = (q!)^{p-r} [(q + 1)!]^r$

Case 2 If $a_p – 1 > a_1$, then by (1) , $a_1 ! \, a_p ! > (a_1 + 1)! \, (a_p – 1)!$

∴ $a_1! \, a_2 ! …a_p ! > (a_1 + 1)! \, a_2 ! …a_{p-1} ! \, (a_p – 1)!$ …. (2)

Rearrange the variables in the R.H.S. of (2) in ascending order and replace the variables by $a_1^{(1)} \le a_2^{(1)} \le … \le a_p^{(1)}$.

∴ We have $a_1! \, a_2 ! …a_p ! > a_1^{(1)} ! \, a_2^{(1)} !…a_p^{(1)} !$.

Repeat the process by increasing the smallest variable by 1 and decreasing the biggest variable by 1.

We get $a_1!\, a_2! \ldots a_p! \;>\; a_1^{(1)}!\, a_2^{(1)}! \ldots a_p^{(1)}! \;>\; \ldots >\; a_1^{(k)}!\, a_2^{(k)}! \ldots a_p^{(k)}!$

After finite number of steps, we get $a_1^{(k)}!\, a_2^{(k)}! \ldots a_p^{(k)}!$ where $a_1^{(k)} = a_p^{(k)} - 1$.

We get Case 1 above and $a_1^{(k)}!\, a_2^{(k)}! \ldots a_p^{(k)}! = (q!)^{p-r}\,[(q+1)!]^r$.

$\therefore\quad a_1!\, a_2! \ldots a_p! \;>\; (q!)^{p-r}\,[(q+1)!]^r$.

10. $$-2 < \frac{3x+11}{x+2} < 2 \Leftrightarrow \begin{cases} -2 < \dfrac{3x+11}{x+2} & \ldots(1) \\ \dfrac{3x+11}{x+2} < 2 & \ldots(2) \end{cases}$$

Solving (1), we get $x < -3$ or $x > -2$ …. (3)

Solving (2), we get $-7 < x < -2$ …. (4)

$\therefore$ The range of values of x is $-7 < x < -3$.

11. $x^2 + 7y^2 + 20z^2 + 8yz - 2zx + 4xy \equiv a(x + py + qz)^2 + b(y + rz)^2 + cz^2$

Equating coefficients of

x^2-term	:		$\therefore\ a = 1$
xy-term	:	$4 = 2pa = 2p$.	$\therefore\ p = 2$
y^2-term	:	$7 = ap^2 + b = 4 + b$	$\therefore\ b = 3$
xz-term	:	$-2 = 2aq = 2q$	$\therefore\ q = -1$
yz-term	:	$8 = 2apq + 2br = -4 + 6r$	$\therefore\ r = 2$
z^2-term	:	$20 = aq^2 + br + c = 13 + c$	$\therefore\ c = 7$

$x^2 + 7y^2 + 20z^2 + 8yz - 2zx + 4xy \equiv (x + 2y - z)^2 + b(y + 2z)^2 + 7z^2$.

Since the given expression is a positive linear combination of complete squares, it is never negative for real values of x, y, x .

12. $x + y + z \ge 3\sqrt[3]{xyz}$ …. (1)

$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \ge 3\sqrt[3]{\frac{1}{x}\frac{1}{y}\frac{1}{z}}$ …. (2)

(1) × (2), $(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \ge 9$ Since $x + y + z = 1$, $\therefore \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \ge 9$.

Equalities holds iff $x = y = z$.

$x + y \ge 2\sqrt{xy}$ …. (3)

$y + z \ge 2\sqrt{yz}$ …. (4)

$z + x \ge 2\sqrt{zx}$ …. (5)

(3) × (4) × (5), $(x + y)(y + z)(z + x) \ge 8xyz$.

Since $x + y + z = 1$, therefore $1 - x = y + z$, $1 - y = z + x$, $1 - z = x + y$

Hence $(1 - x)(1 - y)(1 - z) \geq 8xyz$.

13. $x^2yz = x^2\left(\sqrt{yz}\right)^2 < x^2\left(\frac{y+z}{2}\right)^2 = x^2\left(\frac{1-x}{2}\right)^2 = \frac{1}{4}\left[\sqrt{x(1-x)}\right]^4 < \frac{1}{4}\left[\frac{x+(1-x)}{2}\right]^4 = \frac{1}{64}$

14. **(a)** $\left(a_1^{\,2} + b_1^{\,2} + c_1^{\,2} + d_1^{\,2}\right)\left(a_2^{\,2} + b_2^{\,2} + c_2^{\,2} + d_2^{\,2}\right) - \left(a_1a_2 + b_1b_2 + c_1c_2 + d_1d_2\right)^2$

$= \left(a_1b_2 - a_2b_1\right)^2 + \left(a_1c_2 - a_2c_1\right)^2 + \left(a_1d_2 - a_2d_1\right)^2 + \left(b_1c_2 - b_2c_1\right)^2 + \left(b_1d_2 - b_2d_1\right)^2 + \left(c_1d_2 - c_2d_1\right)^2 \geq 0$

(b) $x = \frac{aR + bR^3}{r^2} > \frac{2\sqrt{(aR)(bR^3)}}{r^2} = 2\sqrt{ab}\left(\frac{R}{r}\right)^2 > 2\sqrt{ab},$ $\quad \because \frac{R}{r} \geq 1$